Functions in Science and in our Daily Lives

The following are some notes on mathematical functions. New terminology is indicated with bold print. The first few paragraphs were lifted (slightly modified) from: Protter, M.H. and C.B. Morrey Jr. (1977) *College Calculus with Analytic Geometry*, Third Edition, Addison-Wesley Publishing Company. Thanks to Christine Brisson for helpful suggestions.

Contents

I. Introduction

In mathematics and many of the physical sciences, simple formulas occur repeatedly. For example, if r is the radius of a circle and A is its area, then:

$$A = \pi r^2$$

If heat is added to an ideal gas in a container of fixed volume, the pressure p and the temperature T satisfy the relation

$$p = a + cT$$

The relationships expressed by these formulas are simple examples of the concept of function. However, it is not essential that a function be associated with a particular formula. As an example, consider the cost in cents of mailing a package via the U.S. Mail. Suppose postal regulations state that the cost is "6 ¢ per gram or fraction thereof". We can construct the following table:

Weight in grams	0 to 1	above 1 up to 2	above 2 up to 3	above 3 up to 4	above 4 up to 5
cost in cents	6	12	18	24	30

This table could be continued until it reaches the maximum weight permitted by postal regulations. To each weight between 0 and the maximum, there corresponds a precise cost. We have here an example of a function relating weight and cost.

As mentioned, there is no algebraic formula that expresses the relationship between weight and cost. This is often the case in the sciences. An experimenter frequently finds by measurement that the numerical value y of some quantity depends in a *unique* way on the measured value x of some other quantity but the experimenter is unable to find a formula that expresses the relationship between x and y. All we have is the set of ordered pairs of x and y values. In such circumstances, the entire connection between x and y is determined by these ordered pairs. The most general definition of a function then is simply a set of ordered pairs. The set of pairs (r,A) obtained from the formula $A=\pi r^2$ is a function as is the set of pairs of weight and cost that conform to the chart above.

Mathematicians tend to think of functions that relate numbers to each other but this is not a necessary part of the concept of a function. A pricing function might assign a number to each product while a ranking of players in a competition can be thought of as a function that assigns an individual to a number (to 1 it assigns the winner, to 2 the person in second place, and so on). A naming function which assigns names to people involves no numbers. The sets of things related by a function are called the **domain** and the **range** of the function. A function assigns exactly one element of the range to each element of the domain. For the price-function just mentioned, the domain includes all products that have a price and the range includes all possible prices. The ranking function described above has numbers in its domain and players in its range. The naming function has people in its domain and names in its range.

QUESTION [1] Let f_A be an age-function, assigning an age to each person. What are the domain and the range of f_A? What are the domain and the range of the postage function described above?

EXAMPLE (Forwards/Progressive) Assimilation is a process whereby an underlying phoneme expresses itself in different ways depending on what precedes it. One can describe the behaviour of the phoneme in terms of a function. The domain of the function includes the set of possible preceding phonemes while the range includes the set of possible allophones of that phoneme.

EXAMPLE In his essay on "Negation", Frege talked about the thoughts that negation combines with and how they relate to the thoughts that result from the combination. Negation thus behaves like a function whose domain is the set of thoughts (or expressible thoughts) and whose range is the set of thoughts.

II. Notation, arguments, values

Functions are often named with letters from the middle of the alphabet. The phrase "function f assigns y to x" can be abbreviated as:

(1) $f(x)=y$

Equivalently, (1) says that y is the **value** of f when **applied** to the **argument** x. (1) is read "f of x equals y". The basic idea of a function then is that the choice of the argument <u>determines</u> the value. This follows from the fact that every element of the domain is assigned exactly one element of the range. Notice, however, that nothing requires every element of the domain to be assigned to a different element of the range. The price function, for example, would assign the same value to all elements in the domain that have the same price. The ranking function on the other hand would assign each number to a different person (assume this was a race).

QUESTION [2] Under what circumstances would different elements in the domain of the age-function, f_A, be assigned the same value?

EXAMPLE Let f_{mother} be a function which assigns to each individual, his/her biological mother. The equation:

$$f_{mother}(\text{Jacob})=\text{Rebecca}$$

says that Rebecca is Jacob's mother.

QUESTION [3] Under which circumstances would two individuals get assigned the same value by f_{mother}? Write an equation that says that f_{mother} assigns the same value to Jacob as it does to Esau.

QUESTION [4] [1]Write two equations that together say that Jacob and Esau are twins. To write these equations use f_{mother} and another function that you define (as well as the names *Jacob* and *Esau*). In defining the second function, remember to say what its domain is and what its range is and in general what value is assigned to each argument.

EXAMPLE Let f_{name} stand for the naming function mentioned above, whose domain includes individuals and whose range includes names. We can then write: f_{name}(John)=*John*. This says that our function assigns to the individual John, the name *John*. Italics indicate that I am mentioning the name, using it to refer to the name itself, not to the individual. Boldface and underline are also used for this purpose.

QUESTION [5] Since elements of the range of negation are also in its domain, we can apply negation twice. Letting f_{neg} stand for the negation function, we can write: $f_{neg}(f_{neg}(x))$, to mean the negation of the negation of x. Using f_{neg} and the variable x, write a statement that says that if you doubly negate, you get back to where you started.

QUESTION [6] Translate the following equation into colloquial English:

$$f_{mother}(f_{mother}(\text{Bill})) = \text{Rita}$$

There are various ways to abbreviate statements about the domain and the range of a function. (2) below illustrates one style:

(2) $g: M \rightarrow C$

(2) says that g is a function whose domain is M and whose range is C. (2) is sometimes read "g **maps** M **onto** C". If M includes all men, and C includes all children, then g is a function which assigns a child to each man.

EXAMPLE The phrase structure rule

$$S \rightarrow NP\ VP$$

says that a VP combines with an NP to yield a sentence. Categorial Grammarians view this in functional terms. A VP is a syntactic function which, when applied to a noun phrase yields a sentence. Letting NP stand for the set of all noun phrases and letting S stand for the set of all sentences and letting vp stand for a particular verb phrase, we have:

$$vp:\ NP \rightarrow S$$

[1]Thanks to Madeline Holler for help on this question.

A vp is a function whose domain includes all noun phrases and whose range includes all sentences.

III. Function-valued functions

There is almost no limitation on what the domain or the range of a function can include, in particular it can include other functions. If the range of a function g consisted of other functions, then g would assign some function to every element of its domain. If x was in the domain, then g(x) would itself be a function.

If g is function whose values are functions, then g(x) is a new function, and if this new function also has x in its domain it is meaningful to write: g(x)(x). This would be the value of g when we apply it to x and then apply the result to x. For clarity, I tend to insert square brackets here: [g(x)](x).

We can use the notation in (2) above to talk about function-valued functions. For example, if f_{dist} is a function that applies to a location to yield a function that applies to another location to yield a distance in miles we could write:

(3) f_{dist}: L → (L → M)

L: locations, M: distances in miles

$[f_{dist}(NY)](NJ) = 100$ "the distance from NY to NJ is 100 miles".

$[f_{dist}(NJ)](NJ) = 0$

Below I have defined another function-valued function based on the operation of addition:

(4) f_+ : N → (N → N)

N: real numbers

for any x,y $[f_+(x)](y) = x+y$

(5)a below is true, while (5)b is false:

(5) a. $[f_+(2)](3)=5$
b. $[f_+(1)](3)=9$

Below are more examples of functions whose range consists of functions.

EXAMPLE A transitive verb combines with an NP to produce a VP. In a previous EXAMPLE, we saw that VPs can be viewed as syntactic functions from noun phrases to sentences. Pursuing this idea, transitive verbs are syntactic functions that apply to noun phrases to produce new functions, namely VPs. Letting tv stand for a particular transitive verb, we have:

tv: NP → (NP → S)

EXAMPLE The information contained in a phone book maps individuals onto their phone numbers. Imagine a data base that contained information for all phones in the world. The data base might be arranged in such a way that each region is mapped onto the phone book for that region. Such a data base is a function:

f_{phone} : $R \rightarrow (I \rightarrow N)$

R: regions, I:individuals N:numbers

If John has a residence in New Jersey and one in New York, it is likely that:
$[f_{phone}(NY)](John) \neq [f_{phone}(NJ)](John)$

QUESTION [7] What is the domain of f_{phone}(Kansas) ? What would it mean if the following equation below was true?

$f_{phone}(Bombay) = f_{phone}(Bermuda)$

EXAMPLE A linguist finds that the shape of the suffix on nouns in a particular language L depends upon the case assigned to the noun. A chart is drawn up showing the suffix used for each case. This chart represents a function whose domain includes the cases of the language and whose range includes the possible suffixes. Over time the linguist discovers several new paradigms depending on whether the nouns are abstract, concrete, human or non-human. This leads to a new function, f_{suffix} whose domain consists of semantic features and whose range consists of functions from cases to suffixes. f_{suffix} is part of the grammar of language L.

EXAMPLE The following function-valued function is based on multiplication:

f_x: $N \rightarrow (N \rightarrow N)$
for any x,y: $[f_x(x)](y)= xy$ (y multiplied by x)
ex. $[f_x(3)](4)= 12$

QUESTION [8] Which two numbers, x and y, make the following true: $[f_+(y)] = [f_x(x)]$?

QUESTION [9] Let $y=[f_+(3)]([f_x(2)](4))$, what number is y?

QUESTION [10] [2] In the previous question, I used brackets and parentheses to indicate what functions were combining with what arguments. One can indicate this as well using a tree. Draw a tree whose root node is labeled: $[f_+(3)]([f_x(2)](4))$, whose terminal nodes consist of f_+,3,f_x,2,4. Intermediate nodes should be labeled as well. The tree should show the function-argument structure of the root node.

QUESTION [11] Using the functions f_+ and f_x, write an equation that says the following: 2 x (2+2)=8

QUESTION [12] Define a function based on the operation of subtraction.

QUESTION [13] Provide a non-mathematical example of a function whose range consists of functions.

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[2] This problem was suggested to me by Graham Horwood.

QUESTION [14] Write a predicate logic statement that says that multiplication is commutative (eg. 4x8=8x4). Use standard predicate logic symbols along with the function f_x and the equal sign. [Omit this exercise, if you are not familiar with predicate logic].

EXAMPLE A tax assessor needs to know precisely what the size of different areas are in his jurisdiction. He uses a computer that has this information. He enters three locations and gets back the area within the triangle defined by those points. The function used looks like this:

g: L → L → (L → Sq))
L: locations Sq: numbers (of square miles)

for any x,y,z: [[g(x)](y)](z) is the square mileage of the area defined by x,y and z.

Note, that g is a function-valued function, where elements in the range are themselves function-valued functions. This can be iterated.

QUESTION [15] What should the value of [[g(NY)](NJ)](NY) be?

Just as there are functions with functions in their range, there are functions with functions in their domain. If k is a function whose domain includes the function g, we could write: k(g). Note, there is a difference between a. and b. below:

(6) a. k(g)(x)
b. k(g(x))

QUESTION [16] If k didn't have functions in its domain which of ((6) a) and ((6) b) would be impossible (we are assuming the convention here that k,g are names of functions)? Explain your answer.

IV. n-place functions

Above we had a price function whose domain included various products and whose range consisted of amounts of money. Applying the function to a product gives the price of the product. Often, a price is fixed based on what the product is as well as on other factors for example the time of year. In this case we would have a function that takes two or more arguments as input. A function that takes two arguments is called a **two-place function**. The notation introduced above is extended to this case, so that:

f(x,y)=z

says that when f takes two arguments, the first being x and the second y, the result is z. And

f_{price}(oranges, summer)= 60¢/pound

is a fancy way to write that oranges cost 60 cents a pound in the summer.

EXAMPLE. It is known that the color of a person's eyes depends on the genes inherited from both parents. Since blue is recessive, a person's eyes are brown unless both parents contribute a gene for blue-eyes. We could describe this correlation as a function whose domain includes pairs of gene types (one contributed by mom and the other by dad) and whose range includes eye colors (blue and brown).

g(brown, blue)=brown g(blue,blue)=blue.
g(blue,brown)=brown g(brown,brown)=brown.

EXAMPLE In order to log on to the computer, you need to type in your name and then your password. If you're in the system and the password is correct, a welcome screen is displayed, otherwise an unwelcome screen is displayed. The information in the computer is a function, f_c whose domain consists of usernames-password pairs and whose range consists of a welcome and an unwelcome message. Note, the arguments of a two-place function are ordered, so that: f_c(Richard, 4$#x) is not necessarily the same as f_c(4$#x, Richard). It could be that the first is the welcome screen and the second is not.

QUESTION [17] Describe what happens when you check your balance in a bank machine in terms of a two or more place function. (this question is a bit open ended, be creative, but be specific, state what the domain and the range of the function is)

EXAMPLE The meaning of *and* combines with two thoughts to yield a new thought that is true if and only if the two thoughts it applied to were true. Letting f_{and} stand for this function and letting T stand for some true thought and F for some false thought we have:

f_{and}(T,T) is true. f_{and}(T,F) is false
f_{and}(F,T) is false f_{and}(F,F) is false

V. Currying

Section III was about function-valued functions and section IV was about two-place functions. It turns out that there is a tight correlation between these two types of functions. A two-place function can be turned into a one-place function with functions in its range and vice versa. For example, the computer login function procedure was described above in terms of a two-place function taking a username and a password to produce a welcome/unwelcome screen. The login procedure could also be described as a function that applies to a username to produce a new function. This new function applies to a password to produce a welcome/unwelcome screen. If I enter my username, then the computer is in a state, describable with a function, where my password will produce welcome, and yours will not. Going the other way now, f_{suffix} was defined above as a one-place function-valued function. It combined with a semantic feature to give a function that combined with a case to give a suffix. Based on that function, we could define a two-place function that takes a pair of a semantic-feature (e.g. +human) and a grammatical case and gives back a suffix.

QUESTION [18] Based on the two-place definition of the meaning of *and* given in an earlier EXAMPLE, describe a one-place function whose range includes functions. Use the "→" notation to state what the domain and the range of the function are. And say in general what value you get for each argument.

QUESTION [19] Redescribe the two-place price function as a one-place function-valued function. Redescribe the two-place eye-color function as a one-place function-valued function.

Just as there are two place functions there are three and more place functions. They are generally called **n-place functions**. And in general, any n-place function can be redefined as a one-place function with functions in its range. For example, if g is a three place function whose arguments are: a person, a number and a location, and whose value is a time then there is a corresponding function-valued function, let's call it g' such that:

$g': P \rightarrow (N \rightarrow (L \rightarrow T))$
P:people N:numbers L: locations T:times

> g' applied to a person produces a function whose domain is numbers and whose range consists of functions from locations to times.

The process of going from an n-place function to a function-valued function is called **currying**. It is also called Shoenfinkelization.

VI. Constant functions, identity functions

While it is part of the definition of a function that each element in the domain is assigned exactly one value in the range (hence we talk of "<u>the</u> price of x", "<u>the</u> age of y" etc), as noted earlier, there is nothing to prevent several elements in the domain from getting assigned to the same element in the range. A price function might assign the same value to apples as it does to oranges, for example. An extreme case of this is a function that assigns the same value to every argument. Such a function is called a **constant function**. Another special kind of function is one in which the domain and range are identical and every element of the domain is assigned to itself. Such a function is called **the identity function**.

QUESTION [20] Why do we say "<u>the</u> identity function" and not "<u>an</u> identity function"?

VII. Partial Functions

The price function defined at the beginning of these notes had a domain of products and a range consisting of numbers: to each product it assigned a price. Now there may be circumstances in which some products do not have a price, for example, during the winter when there are no cherries in the store it makes no sense to talk about the price of cherries. In this case we want to say there is no value for the price function when applied to cherries (or for the two-place f_{price} we want to say that f_{price}(cherries, winter) is undefined). In other words, the function doesn't actually assign a value to every element in its domain. Such functions are called **partial functions**. Of course, being a function, elements in the domain that are assigned a value would be assigned just one value. A function that assigns values to all elements in its domain is called a **total function**.

EXAMPLE A verb phrase can be thought of as a syntactic function whose domain is NP (the set of noun phrases) and whose range is S (the set of sentences). But due to selectional restrictions not every combination of a noun phrase and verb phrase leads to a grammatical sentence. For example, the verb phrase *lives in New Mexico* doesn't assign any element of S to the noun phrase *Monday*.[3] And for perhaps other reasons, it doesn't assign anything to the

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[3] What I'm saying here is that *Monday lives in New Mexico* is not a sentence of English. "is a sentence of English" is being construed broadly here. This suffices for the purposes at hand, though being more careful we might want to say that *Monday lives in New Mexico* is a grammatical sentence of English which is semantically anomalous.

noun phrase *himself*. The verb phrase *lives in New Mexico* is a partial function from noun phrases to sentences.

EXAMPLE The two place function f that maps a pair of real numbers *a,b* into the quotient of *a* divided by *b* (e.g. f(6,2)=3, f(5,2)=2.5) is not defined when b=0. So it is a partial function whose domain consists of pairs of real numbers.

A partial function could be "totalized" by redefining the domain leaving out elements that aren't assigned a value. Similarly, a total function could be partialized by redefining the domain adding elements that aren't assigned a value.

QUESTION [21] Provide an example of a morphological function that is partial. By morphological function, I mean a function that applies to morphemes from a certain syntactic category and produces new forms (whether they are words, sequences of morphemes, or new simple morphemes). Try to think of one example where the domain and range are the same (e.g. it maps nouns into nouns) and one in which they are different. Say why the function is partial, i.e. what elements of the domain are not assigned a value.

VIII. Characteristic Functions

The postage function introduced earlier corresponded to a chart pairing weights and cost. Similarly, there are functions that correspond to simple lists. For example, if I draw up a list of the students in the department who are taking this course, in effect, I am pairing students with the values "in (the class)" and "out" or with the values "on (the list)" and "off". Any function whose range consists of just two values corresponds to a list of things in the domain, or to a **set** of things in the domain. Since these functions characterize sets they are called **characteristic** functions.

EXAMPLE $k: P \rightarrow N$
P:people N: the numbers 1 and 0.
k(Zsusza)=1, k(Madeline)=1, for all other people, k(x)=0.

k characterizes the set consisting of Zsuzsa and Madeline. It is customary to use 1 for "in" and 0 for "out".

QUESTION [22] Define a function that characterizes the set containing female faculty members in the Linguistics Department. Give the domain, the range and state how the mapping works, as in the previous example.

QUESTION [23] f_p is a function that characterizes a set containing anything and everything that is red and nothing else. g_k is a function that characterizes another set which we will call K. The two functions are related as follows:
For any x, if $g_k(x) = 0$ then $f_p(x) = 1$ and
if $g_k(x) = 1$ then $f_p(x) = 0$.

What does K contain?

QUESTION [24] Let f_p, f_b, f_r, and f_q be characteristic functions. Below is a list showing what set each of these functions characterizes:

f_p: a set containing anything and everything that is red and nothing else
f_b: a set containing anything and everything that is broken and nothing else
f_r: a set containing all broken things and all red things (including broken red things) and nothing else.
f_q: a set containing anything that is both red and broken and nothing else

The range of all of these functions contains two values: 0 and 1. The domain of all these functions contains any physical object, whatsoever. The last two functions are related to the first two. If you know what values f_p and f_b assign to an object, you can figure out what f_r and what f_q will assign. State exactly what the relation is in a way similar to the way g_k and f_p were related in the previous question.

IX. Lambda Operator

The **lambda operator** can be added to predicate logic to produce function naming expressions. Like the quantifiers, it attaches to formulae. The following are examples of lambda expressions:

λxFx $\lambda x(Fx \wedge Gx)$ $\lambda x(Fx \rightarrow Gx)$ $\lambda x(Fx \rightarrow Gx)$

$\lambda x\forall yRxy$ $\lambda y\exists xRxy$

Unlike the quantifiers, however, these expressions are not themselves formulae. They name functions. The range of these functions consists of truth-values (true and false).

EXAMPLE Assuming the key below:

Fx	:	x is a frog
Gx	:	x giggles
Axy	:	x ate y
Lxy	:	x lifted y
j	:	John
k	:	Kermit
c	:	my coffee cup

We can use λxFx to name the function that assigns true to something if it is a frog and false otherwise. The function $\lambda x(Gx \vee Fx)$ will assign true to something, say John, just in case he giggles or is a frog. $\lambda x\neg Fx$ assigns true to something if it is not a frog, and otherwise it assigns false. $\lambda xLxx$ names a function that would apply to Jeremiah to give true if Jeremiah lifted himself and false otherwise. $\lambda xLkx$ names a function that would apply to Jeremiah to give true if Kermit lifted Jeremiah and false otherwise.

Lambda-expressions can be used like other names of functions. We can, for example, apply them to arguments: $\lambda x(Fx \vee Gx)(j)$.

EXAMPLES $\lambda xFx(k)$ is true just in case Fk is true, in other words, if Kermit is a frog. $\lambda x(Fx \vee Gx)(j)$ is true if $(Fj \vee Gj)$ is true. $\lambda x\exists yAxy(j)$ is true just in case John ate something. $\lambda y\exists xAxy(c)$ is true just in case my coffee cup was eaten.

QUESTION [25] Assuming the following key:

Hxy : x likes y's house

What conditions would have to hold if the function $\lambda xHxx$ applied to John to give true?

Sentence (7) below is ambiguous.

(7) John likes his house and Kermit does too.

On the so-called sloppy reading, Kermit likes his own house, whereas on the strict reading, Kermit likes John's house. One can think of the two readings in terms of two different functions. According to (7) on the sloppy reading, the function $\lambda xHxx$ applies to John to give true and it also applies to Kermit to give true. According to (7), on the strict reading, there is a function which applies to John to give true and it also applies to Kermit to give true. Write a lambda expression to name this second function.

This section has illustrated the most basic use of the lambda operator. Its use can be extended syntactically (eg. attaching to things that aren't formula) and semantically (eg. to name functions with different domains and ranges).

X. Monotonic Increasing, Monotonic Decreasing

In developing economic theories, use is made of the following two functions:

f_1: Output → Production Cost

Output is a measure of how much/many goods are produced (e.g. 14 tons of cotton per year)
Production Cost is the amount of money invested in producing the goods.

f_2: SellingPrice → Demand

Demand is a measure of the quantity of the goods demanded by consumers (how you measure that, I don't know)

In the first case, since the higher the output the more the production cost, the value of f_1 goes up as the argument goes up. In the second case, since the demand usually goes down when the price goes up, the value of f_2 goes down as the argument goes up.

The first kind of function is called an **increasing** function and the second is called a **decreasing** function. A function f is increasing, if $f(x) \succ f(y)$, for any x,y in its domain where $x \succ y$ (read "x is greater than y"). A function f is decreasing, if $f(x) \prec f(y)$, for any x,y in its domain where $x \succ y$. Since the movement is steadily up or down, sometimes the terms **monotone increasing** and **monotone decreasing** are used.

These terms make sense when the domains and ranges of the functions are numbers because then there is a natural way to understand what it means to go up or down. However, the terms can be used for other kinds of functions, as long as one defines what it means to go up or down. Here are two examples.

EXAMPLE The following function assigns to a group of individuals the number of individuals in that group:

g_1: G → N

G: all possible groups of individuals

N: whole numbers

For example, if c was a group consisting of Peter, Paul and Mary, then g(c)=3. Clearly, a group always has more members then any of its proper subgroups. If we think of any group as being higher on a scale than its subgroups, then this fact translates into a claim that g_1 is monotonically increasing. If we applied g to a group of all American folk musicians, including Peter, Paul and Mary, we would be applying it to an argument that is greater than c, and so the result would be a higher number than g(c).

EXAMPLE Above we defined a function, f_{neg}, whose domain and range consisted of thoughts. $f_{neg}(x)$ is the negation of x. One can define a scale for thoughts based on the notion of entailment. If A entails B, then A is more informative than B, so we will say that A is 'higher' on the entailment scale than B. Now, consider the following example which illustrates a property of negation. If John owns a red car, then John owns a car. Conversely, if John doesn't own a car, then John doesn't own a red car. In general, if A entails B, then the negation of B entails the negation of A. This means that $f_{neg}(B) \succ f_{neg}(A)$ for any A,B where $A \succ B$, where by "greater-than" we mean in terms of our entailment based scale. In other words, given this way of ordering thoughts (=propositions), f_{neg} is monotonically decreasing.

QUESTION [26] Consider a function f_{might}, whose domain and range consists of thoughts, where if thought A is expressed by sentence S, $f_{might}(A)$ would be expressed by *It might be that* S. So applying f_{might} to the thought expressed by *Akin comes from Nigeria* gives you the thought expressed by *It might be that Akin comes from Nigeria*. Taking entailment to provide a scale for thoughts, as in the previous example, is f_{might} monotonically increasing? monotonically decreasing? non-monotonic? Justify your answer with examples.